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Tentamen HT08
Sannolikhet och statistik
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Uppgift 1

a) Sannolikhetsfunktion:

$$p_X(k) = c \cdot k \quad \text{med} \quad \sum_{k=1}^{6} p_X(k) = 1 \quad \Longrightarrow \quad c = \frac{1}{21}$$

k	1	2	3	4	5	6
$p_X(k)$	1/21	2/21	3/21	4/21	5/21	6/21

b) Fördelningsfunktion för X:

$$F_x(k) = \sum_{i=1}^{k} p_X(i)$$

k	1	2	3	4	5	6
$F_X(k)$	1/21	1/7	2/7	10/21	5/7	1

c) Väntevärde, varians:

$$\begin{aligned}
E(X) &= 1 \cdot 1/21 + 2 \cdot 2/21 + 3 \cdot 3/21 + 4 \cdot 4/21 + 5 \cdot 5/21 + 6 \cdot 6/21 = 13/3 \approx 4.33 \\
E(X^2) &= 1^2 \cdot 1/21 + 2^2 \cdot 2/21 + 3^2 \cdot 3/21 + 4^2 \cdot 4/21 + 5^2 \cdot 5/21 + 6^2 \cdot 6/21 = 21 \\
V(X) &= E(X^2) - E(X)^2 = 21 - \frac{13^2}{3^2} = \frac{20}{9}
\end{aligned}$$

d) Sannolikhet $P(A)$ att kasta ett jämnt tal:

$P(A) = P\left[(k=2) \cup (k=4) \cup (k=6)\right] = 2/21 + 4/21 + 6/21 = 12/21$ k=2, k=4, k=6 oförenliga

e) En person kastar tärningen 10 gånger ...

$$\begin{aligned}
Y &= \sum_{i=1}^{10} X_i \\
E(Y) &= E\left(\sum_{i=1}^{10} X_i\right) = \sum_{i=1}^{10} E\left(X_i\right) = \sum_{i=1}^{10} \frac{13}{3} = \frac{130}{3} \\
V(Y) &= V\left(\sum_{i=1}^{10} X_i\right) = (oberoende) = \sum_{i=1}^{10} V\left(X_i\right) = \sum_{i=1}^{10} \frac{20}{9} = \frac{200}{9} \\
D(Y) &= \sqrt{V(Y)} = \sqrt{\frac{200}{9}} = \sqrt{2} \cdot \frac{10}{3}
\end{aligned}$$

Uppgift 2

P(diabetes) = P(D) ; P(kvinna) = P(K) ; P(man) = P(M)

Kontingenstabell:

	K	M	
D	0.006	0.02	0.026
D^*	0.594	0.38	0.974
	0.6	0.4	1

a) P(D) ? **Lagen om total sannolikhet**

$$\begin{aligned} P(D) &= P(D|M)\cdot P(M) + P(D|K)\cdot P(K) \\ &= 0.05\cdot 0.4 + 0.01\cdot 0.6 = 0.026 \qquad \text{(lika med den marginala slh:en i råd 1)} \end{aligned}$$

b) K och D oberoende?

$$P(D) = 0.026 \quad P(D|K) = 0.01 \quad \text{dvs.} \quad P(D) \neq P(D|K) \quad \Longrightarrow \quad \text{beroende}$$

c) P(K|D); P(M|D) ? **Bayes sats**!

$$\begin{aligned} P(K|D) &= \frac{P(D|K)P(K)}{P(D)} = \frac{0.01\cdot 0.6}{0.026} = \frac{6}{26} \\ P(M|D) &= \frac{P(D|M)P(M)}{P(D)} = \frac{0.05\cdot 0.4}{0.026} = \frac{20}{26} \end{aligned}$$

Uppgift 3

$$P(X > 15) = \int_{15}^{\infty} f_X(t)\, dt = \int_{15}^{\infty} 0.1\cdot e^{-0.1\cdot t}\, dt = \left[-e^{-0.1t}\right]_{15}^{\infty} = e^{-1.5} \approx 22.31\%$$

Uppgift 4

a) $X \in N(3, 0.15)$

$$P(2.85 < X \leq 3.15) = F_X(3.15) - F_X(2.85) = \Phi\left(\frac{3.15-3}{0.15}\right) - \Phi\left(\frac{2.85-3}{0.15}\right) = \Phi(1) - \Phi(-1) = 2\Phi(1) - 1 \approx 68.3\%$$

b)

$$P(X > 3.3) = 1 - P(X \leq 3.3) = 1 - F_X(3.3) = 1 - \Phi\left(\frac{3.3-3}{0.15}\right) = 1 - \Phi(2) \approx 2.3\%$$

c)

$$Y = \sum_{i=1}^{100} X_i \quad \text{med} \quad X_i \in N(3, 0.15) \quad \Rightarrow \quad Y \in N\left(100\cdot 3, \sqrt{100}\cdot 0.15\right) \quad \text{dvs.} \quad Y \in N(300, 1.5)$$

$$P(Y > 303) = 1 - P(Y \leq 303) = 1 - F_Y(303) = 1 - \Phi\left(\frac{303-300}{1.5}\right) = 1 - \Phi(2) \approx 2.3\%$$

d) $n = 25$

$$\bar{X} \in N\left(\mu, \sigma/\sqrt{n}\right) \quad \Rightarrow \quad \bar{X} \in N(3, 0.03)$$

$$P\left(\bar{X} > 3.03\right) = 1 - P\left(\bar{X} \leq 3.03\right) = 1 - F_{\bar{X}}(3.03) = 1 - \Phi\left(\frac{3.03-3}{0.03}\right) = 1 - \Phi(1) \approx 15.9\%$$

Uppgift 5

a) $X \in Bin(n,p)$; $X \in Bin(n, 0.05)$

b) $n = 10$

$$\begin{aligned} p_X(k) &= \binom{n}{k} p^k (1-p)^{n-k} \\ p_X(0) &= 0.599 \\ p_X(1) &= 0.315 \\ p_X(2) &= 0.075 \end{aligned}$$

eller

$$F_x(2) = 0.9885 \quad \text{tabell över binomialfördelning med } n = 10;\ p = 0.05;\ x = 2$$

c)

$$P(X > 2) = 1 - P(X \leq 2) = 1 - F_X(2) = 1 - 0.9885 \approx 1.15\%$$

d) $X \in Bin(1000, 0.05)$

$$\begin{aligned} E(X) &= n \cdot p = 1000 \cdot 0.05 = 50 \\ V(X) &= n \cdot p \cdot (1-p) = 50 \cdot 0.95 = 47.5 \\ D(X) &= \sqrt{V(X)} = \sqrt{47.5} \approx 6.89 \end{aligned}$$

e) $X \in Bin(1000, 0.05)$ med $n \cdot p \cdot (1-p) > 10 \quad \Rightarrow \quad X \in AsN(50, 6.89)$

$$P(X > 30) = 1 - P(X \leq 30) = 1 - F_X(30) = 1 - \Phi\left(\frac{30.5 - 50}{6.89}\right) = 1 - \Phi(-2.83) \approx 99\% \quad \text{(halvkorrektion)}$$

Uppgift 6

a)

$$I_\mu = \bar{x} \pm t_{\alpha/2}(n-1) \cdot d \quad \text{med} \quad d = s_x/\sqrt{n}$$

$$\bar{x} = 2.31 \quad s_x = 0.225 \quad d = 0.1 \quad t_{0.025}(4) = 2.78$$

$$I_\mu = 2.31 \pm 0.28$$

b)

$$I_{\Delta\mu} = \bar{x} - \bar{y} \pm t_{\alpha/2}(f) \cdot d$$

$$f = n_x + n_y - 2 \qquad s^2 = \frac{(n_x - 1)s_x^2 + (n_y - 1)s_y^2}{f} \qquad d = s\sqrt{1/n_x + 1/n_y}$$

$$f = 9 \qquad s = 0.217 \qquad d = 0.1314 \qquad t_{0.025}(9) = 2.26$$

$$I_{\Delta\mu} = 0.2 \pm 0.3 = (-0.1, 0.5)$$

c) **Parade observationer!**

Person	1	2	3	4	5
före	2.25	2.50	1.95	2.40	2.40
efter	2.40	2.70	2.20	2.55	2.65
z_i	0.15	0.2	0.25	0.15	0.25

$$I_{\Delta\mu} = \bar{z} \pm t_{\alpha/2}(f) \cdot d \qquad d = s_z/\sqrt{n} \qquad f = n - 1$$

$$\bar{z} = 0.2 \quad s_z = 0.05 \quad d = 0.022 \quad t_{0.025}(4) = 2.78$$

$$I_{\Delta\mu} = 0.2 \pm 0.06 = (0.14, 0.26)$$

Uppgift 7

a) $y_i = \alpha + \beta x_i + \epsilon_i$

b)

$$\beta^* = \frac{S_{xy}}{S_{xx}} \qquad S_{xy} = \sum_{i=1}^{n} x_i y_i - n\bar{x}\bar{y} \qquad S_{xx} = \sum x_i^2 - n\bar{x}^2 \quad \Rightarrow \quad \beta^* = \frac{2.95}{0.043} = 68.6$$

c) 95% konfidensintervall för β

$$I_\beta = \beta^* \pm t_{\alpha/2}(n-2) \cdot d$$

$$d = \frac{s}{\sqrt{S_{xx}}} \qquad s = \sqrt{Q_0/(n-2)} \qquad Q_0 = S_{yy} - S_{xy}^2/S_{xx}$$

$$Q_0 = 210 - 2.95^2/0.043 = 7.62 \quad s = 1.593 \quad d = 7.684 \quad t_{0.025}(3) = 3.18$$

$$I_\beta = 68.6 \pm 24.4 \approx (44, 93)$$